STRUCTURAL PROPERTIES AND CLASSIFICATION OF REES MATRIX SEMIGROUPS WITH APPLICATIONS TO COMPLETELY SIMPLE SEMIGROUPS

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ABSTRACT

Rees matrix semigroups constitute one of the principal structural constructions in semigroup theory. They describe a broad class of semigroups by combining a group, two nonempty index sets, and a sandwich matrix whose entries determine the interaction among the constituent group components. This paper presents a systematic study of the construction, algebraic properties, and classification of Rees matrix semigroups over groups. Particular attention is given to associativity, idempotents, regularity, maximal subgroups, Green's relations, principal ideals, normalization of sandwich matrices, and isomorphism criteria. The Rees–Suschkewitsch theorem is formulated and discussed as the central classification theorem for completely simple semigroups. It is shown that every completely simple semigroup is isomorphic to a Rees matrix semigroup over one of its maximal subgroups, while the row and column indices encode its minimal right and left ideals. The extension to Rees zero-matrix semigroups and completely 0-simple semigroups is also considered. Applications to minimal ideals of finite semigroups, semigroup representations, semigroup algebras, generating sets, rank calculations, and finite classification problems are presented. The study demonstrates that the Rees construction reduces many structural questions concerning completely simple semigroups to explicit computations involving groups, index sets, and sandwich matrices.

Keywords: Rees matrix semigroup, completely simple semigroup, sandwich matrix, Green's relations, maximal subgroup, idempotent, semigroup classification.

1. INTRODUCTION

A semigroup is an algebraic system equipped with an associative binary operation. Although its definition differs from that of a group only through the absence of an identity element and inverses, the structure of a general semigroup can be considerably more complicated than the structure of a group. Semigroup theory consequently seeks structural decompositions that identify group-like components and describe how these components interact.

One of the most important advances in this direction was made by D. Rees, who introduced matrix semigroups over groups and used them to describe simple and 0-simple semigroups [1]. The resulting Rees theorem is now regarded as a foundational structure theorem: completely simple semigroups are precisely the Rees matrix semigroups over groups. Rather than representing elements by ordinary numerical matrices, the Rees construction represents them by triples consisting of a row index, a group element, and a column index. Multiplication is controlled by a sandwich matrix.

The structural interpretation of the construction became particularly transparent after the introduction of Green's relations [2]. These equivalence relations classify semigroup elements according to the principal left, right, and two-sided ideals they generate. In a Rees matrix semigroup, Green's relations can be read directly from the row and column coordinates. The maximal subgroups are the intersections of corresponding row and column classes, and each such subgroup is isomorphic to the underlying group.

Clifford developed matrix representations of completely simple semigroups and clarified the connection between the Rees construction and classical matrix methods [3]. Subsequent systematic accounts established Rees matrix semigroups as indispensable objects in the study of regular semigroups, minimal ideals, principal factors, representations, and semigroup algebras [4], [5]. More abstract characterizations have shown that the Rees construction is not merely a convenient coordinate system; it reflects intrinsic relationships between elements, ideals, and idempotents [6].

The present paper has four principal objectives. First, it develops the fundamental construction of Rees matrix semigroups and verifies its algebraic properties. Second, it determines Green's relations, idempotents, inverses, maximal subgroups, and ideal structure. Third, it presents normalization and isomorphism criteria for the classification of Rees matrix semigroups. Finally, it explains how the Rees–Suschkewitsch theorem applies these results to completely simple semigroups and related finite-semigroup problems.

2. PRELIMINARY CONCEPTS

Let S be a nonempty set equipped with an associative binary operation. Then S is a semigroup when

$$(xy)z = x(yz)$$

for all $x, y, z \in S$.

An element $e \in S$ is called an idempotent when

$$e^2 = e.$$

An element $a \in S$ is regular if there exists $x \in S$ such that

$$axa = a.$$

If every element of S is regular, then S is called a regular semigroup. If x also satisfies

$$xax = x,$$

then x is an inverse of a. Unlike a group inverse, an inverse in a regular semigroup need not be unique.

A semigroup is simple if it has no proper nonempty two-sided ideal. An idempotent e is primitive if the conditions

$$f^2 = f, \qquad ef = fe = f$$

for an idempotent f imply $f = e$. A completely simple semigroup is a simple semigroup containing a primitive idempotent. Completely simple semigroups are regular and can be expressed as unions of mutually isomorphic maximal subgroups [4], [5].

For a semigroup S, let S^1 denote S with an identity adjoined when it does not already possess one. Green's relations are defined by

$$a\, L\, b \Leftrightarrow S^1 a = S^1 b,$$

$$a\,R\,b \Leftrightarrow aS^1 = bS^1,$$

and

$$a\,J\,b \Leftrightarrow S^1aS^1 = S^1bS^1.$$

The remaining principal relations are

$$H = L \cap R$$

and

$$D = L \circ R = R \circ L.$$

An H-class containing an idempotent is a maximal subgroup of the semigroup. Green's relations therefore provide the natural framework for locating group components inside regular semigroups [2].

3. CONSTRUCTION OF REES MATRIX SEMIGROUPS

Let G be a group with identity 1_G. Let I and Λ be nonempty sets, and let

$$P = (p_{\lambda i})$$

be a $\Lambda \times I$ matrix whose entries belong to G. The matrix P is called the sandwich matrix.

The Rees matrix semigroup over G, with index sets I and Λ and sandwich matrix P, is denoted by

$$M[G; I, \Lambda; P].$$

Its underlying set is

$$I \times G \times \Lambda.$$

Multiplication is defined by

$$(i, g, \lambda)(j, h, \mu) = \big(i, gp_{\lambda j}h, \mu\big).$$

The first coordinate comes from the first factor, the third from the second factor, and the group coordinate is modified by the matrix entry $p_{\lambda j}$.

3.1 Associativity

Let

$$a = (i, g, \lambda), \qquad b = (j, h, \mu), \qquad c = (k, t, \nu).$$

Then

$$(ab)c = \big(i, gp_{\lambda j}h, \mu\big)(k, t, \nu),$$

and hence

$$(ab)c = \big(i, gp_{\lambda j}hp_{\mu k}t, \nu\big).$$

On the other hand,

$$a(bc) = (i, g, \lambda)\big(j, hp_{\mu k}t, \nu\big),$$

so that

$$a(bc) = \big(i, gp_{\lambda j}hp_{\mu k}t, \nu\big).$$

Therefore,

$$(ab)c = a(bc),$$

and the multiplication is associative.

The sandwich matrix is essential because it records how the group component associated with one row-column position interacts with the component associated with another position. When every matrix entry is 1_G, the multiplication reduces to

$$(i, g, \lambda)(j, h, \mu) = (i, gh, \mu).$$

This special case is called a rectangular group.

4. STRUCTURAL PROPERTIES

4.1 Idempotents

Let $a = (i, g, \lambda)$. Then

$$a^2 = (i, gp_{\lambda i}g, \lambda).$$

Thus a is idempotent precisely when

$$gp_{\lambda i}g = g.$$

Multiplying by g^{-1} gives

$$p_{\lambda i}g = 1_G,$$

and therefore

$$g = p_{\lambda i}^{-1}.$$

Consequently, the idempotents of the Rees matrix semigroup are exactly

$$E(M[G; I, \Lambda; P]) = \{(i, p_{\lambda i}^{-1}, \lambda) : i \in I,\ \lambda \in \Lambda\}.$$

There is exactly one idempotent in every set having fixed first and third coordinates.

4.2 Regularity and Inverses

Every Rees matrix semigroup over a group is regular. Let

$$a = (i, g, \lambda).$$

Choose arbitrary $j \in I$ and $\mu \in \Lambda$, and define

$$b = (j, p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1}, \mu).$$

Then

$$aba = a$$

and

$$bab = b.$$

Indeed, the middle coordinate of aba is

$$gp_{\lambda j}(p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1})p_{\mu i}g = g,$$

while the corresponding calculation for bab produces the middle coordinate of b. Since j and μ may be selected arbitrarily, an element generally possesses several inverses.

4.3 Green's Relations

Let

$$a = (i, g, \lambda) \quad and \quad b = (j, h, \mu).$$

Their Green classes are determined entirely by their outer coordinates:

$$a\,R\,b \Leftrightarrow i = j,$$

$$a\,L\,b \Leftrightarrow \lambda = \mu,$$

and consequently,

$$a\,H\,b \Leftrightarrow i = j \quad and \quad \lambda = \mu.$$

Thus the R-classes are

$$R_i = \{i\} \times G \times \Lambda,$$

while the L-classes are

$$L_\lambda = I \times G \times \{\lambda\}.$$

Their intersections are

$$H_{i\lambda} = \{i\} \times G \times \{\lambda\}.$$

Every R-class intersects every L-class. Hence the entire semigroup forms a single D-class. It is also a single J-class because every element generates the whole semigroup as a two-sided ideal:

$$M[G; I, \Lambda; P]\,a\,M[G; I, \Lambda; P] = M[G; I, \Lambda; P].$$

This proves that a Rees matrix semigroup over a group is simple.

4.4 Maximal Subgroups

The unique idempotent in $H_{i\lambda}$ is

$$e_{i\lambda} = \left(i, p_{\lambda i}^{-1}, \lambda\right).$$

The H-class $H_{i\lambda}$ is a group with identity $e_{i\lambda}$. Define

$$\phi_{i\lambda}: H_{i\lambda} \to G$$

by

$$\phi_{i\lambda}(i, g, \lambda) = p_{\lambda i} g.$$

For $g, h \in G$,

$$\phi_{i\lambda}\big((i, g, \lambda)(i, h, \lambda)\big) = p_{\lambda i} g p_{\lambda i} h.$$

This is equal to

$$\phi_{i\lambda}(i, g, \lambda)\phi_{i\lambda}(i, h, \lambda).$$

Therefore, $\phi_{i\lambda}$ is a group isomorphism. Every maximal subgroup of the Rees matrix semigroup is consequently isomorphic to G.

4.5 Rectangular-Band Quotient

The group coordinates may be removed through the mapping

$$\rho(i, g, \lambda) = (i, \lambda).$$

The image is the rectangular band $I \times \Lambda$ under the multiplication

$$(i, \lambda)(j, \mu) = (i, \mu).$$

Since

$$\rho\big((i, g, \lambda)(j, h, \mu)\big) = (i, \mu),$$

the map ρ is a surjective homomorphism. A Rees matrix semigroup may therefore be interpreted as a rectangular arrangement of copies of the group G, with the sandwich matrix governing multiplication between the copies.

5. Normalization and Isomorphism Classification

Different sandwich matrices may determine isomorphic semigroups. Classification therefore requires identifying the transformations of P that do not change the isomorphism type.

5.1 Normalization of the Sandwich Matrix

Choose fixed indices

$$i_0 \in I \quad and \quad \lambda_0 \in \Lambda.$$

The sandwich matrix can be replaced by an equivalent normalized matrix $Q = (q_{\lambda i})$, where

$$q_{\lambda i} = p_{\lambda i_0}^{-1} p_{\lambda i} p_{\lambda_0 i}^{-1} p_{\lambda_0 i_0}.$$

For the distinguished column,

$$q_{\lambda i_0} = 1_G,$$

and for the distinguished row,

$$q_{\lambda_0 i} = 1_G.$$

Thus every Rees matrix semigroup is isomorphic to one whose selected row and column consist entirely of the identity element. Normalization removes redundant information arising from independent multiplication of rows and columns by group elements.

5.2 Isomorphism Criterion

Consider two Rees matrix semigroups

$$S = M[G; I, \Lambda; P]$$

and

$$T = M[H; J, \Gamma; Q].$$

They are isomorphic if and only if there exist:

1. a group isomorphism

$$\theta: G \to H;$$

2. bijections

$$\alpha: I \to J \qquad and \qquad \beta: \Lambda \to \Gamma;$$

3. elements $u_i \in H$ for every $i \in I$, and $v_\lambda \in H$ for every $\lambda \in \Lambda$;

such that

$$q_{\beta(\lambda),\alpha(i)} = v_\lambda\, \theta(p_{\lambda i})\, u_i$$

for all $i \in I$ and $\lambda \in \Lambda$.

Under these conditions, an isomorphism is given by

$$\Phi(i, g, \lambda) = \left(\alpha(i), u_i^{-1}\theta(g)v_\lambda^{-1}, \beta(\lambda)\right).$$

To verify multiplicativity, let

$$x = (i, g, \lambda), \qquad y = (j, h, \mu).$$

Then the middle coordinate of $\Phi(x)\Phi(y)$ is

$$u_i^{-1}\theta(g)v_\lambda^{-1}q_{\beta(\lambda),\alpha(j)}u_j^{-1}\theta(h)v_\mu^{-1}.$$

Substitution of the matrix relation gives

$$u_i^{-1}\theta(g)\theta\left(p_{\lambda j}\right)\theta(h)v_\mu^{-1},$$

which is exactly the middle coordinate of $\Phi(xy)$.

The criterion shows that the isomorphism class is determined not by the individual entries of P, but by the orbit of P under group automorphisms, permutations of rows and columns, and row-column scaling. Automorphisms of a given Rees matrix semigroup arise from the same data when $G = H$, $I = J$, $\Lambda = \Gamma$, and the transformed matrix is equivalent to P.

6. THE REES–SUSCHKEWITSCH THEOREM

The fundamental classification theorem may be stated as follows.

Theorem 1 (Rees–Suschkewitsch Theorem). A semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup over a group [1], [4], [5].

PROOF OUTLINE

The reverse direction follows from the structural results already obtained. A Rees matrix semigroup over a group is simple, regular, and contains primitive idempotents. It is therefore completely simple.

For the forward direction, let S be a completely simple semigroup. Choose an idempotent $e \in S$, and let $G = H_e$ be the maximal subgroup containing e. Let I index the R-classes of S, and let Λ index its L-classes.

Appropriate representatives are selected from each R-class and each L-class. Every element of S can then be expressed uniquely in a form corresponding to

$$(i, g, \lambda),$$

where $g \in G$. Products of the selected representatives determine group elements $p_{\lambda i}$, producing a sandwich matrix

$$P = (p_{\lambda i}).$$

Multiplication in S then becomes

$$(i, g, \lambda)(j, h, \mu) = \left(i, gp_{\lambda j}h, \mu\right).$$

The resulting correspondence is an isomorphism from S onto

$$M[G; I, \Lambda; P].$$

The theorem identifies the three essential components of a completely simple semigroup. The group G records the structure of every maximal subgroup, I indexes the minimal right ideals, Λ indexes the minimal left ideals, and P records how these components are linked.

7. REES ZERO-MATRIX SEMIGROUPS

The Rees construction extends to semigroups with zero. Let

$$G^0 = G \cup \{0\},$$

where multiplication involving 0 produces 0. Let P be a matrix over G^0. The Rees zero-matrix semigroup is

$$M^0[G; I, \Lambda; P] = (I \times G \times \Lambda) \cup \{0\}.$$

Its multiplication is defined by

$$(i, g, \lambda)(j, h, \mu) = \{(i, gp_{\lambda j}h, \mu), p_{\lambda j} \neq 0, 0, p_{\lambda j} = 0,$$

together with

$$0x = x0 = 0.$$

The sandwich matrix is called regular when every row and every column contains at least one nonzero entry. In that case, the Rees zero-matrix semigroup is completely 0-simple. Conversely, every completely 0-simple semigroup is isomorphic to a regular Rees zero-matrix semigroup over a group [1], [5].

Zeros in the sandwich matrix record forbidden products between particular L- and R-classes. The associated bipartite graph has vertex set $I \cup \Lambda$, with an edge between i and λ whenever

$$p_{\lambda i} \neq 0.$$

8. EXAMPLES

8.1 Groups

Take singleton index sets

$$I = \{1\}, \qquad \Lambda = \{1\},$$

and let

$$P = (1_G).$$

Then

$$M[G; I, \Lambda; P]$$

is naturally isomorphic to G.

8.2 Rectangular Bands

Let G be the trivial group. The resulting Rees matrix semigroup is the rectangular band

$$I \times \Lambda$$

with multiplication

$$(i, \lambda)(j, \mu) = (i, \mu).$$

Thus rectangular bands are precisely the completely simple semigroups whose maximal subgroups are trivial.

8.3 Rectangular Groups

If every entry of P is 1_G, then

$$(i, g, \lambda)(j, h, \mu) = (i, gh, \mu).$$

The semigroup is isomorphic to the direct product of G and a rectangular band.

8.4 A Nontrivial Sandwich Matrix

Let

$$G = C_2 = \{1, a\}, \qquad a^2 = 1,$$

and take

$$I = \Lambda = \{1,2\},$$

with

$$P = (1\ 1\ 1\ a\ ).$$

The resulting semigroup has eight elements and four idempotents:

$$(1,1,1), \quad (2,1,1), \quad (1,1,2), \quad (2, a, 2).$$

Although each maximal subgroup is isomorphic to C_2, the entry a produces a nontrivial interaction among the group components. This example demonstrates that the maximal subgroup and the numbers of L- and R-classes do not by themselves determine the semigroup; the sandwich matrix is an indispensable invariant.

9. APPLICATIONS

9.1 Minimal Ideals of Finite Semigroups

Every nontrivial finite semigroup contains idempotents, and its minimal ideal is generally the most group-like part of its structure. A nonzero minimal ideal of a finite semigroup is completely simple and can therefore be represented as

$$M[G; I, \Lambda; P].$$

Consequently, the internal structure of the minimal ideal can be studied through a finite group, finite index sets, and a finite sandwich matrix. This principle is frequently used in the analysis of transformation semigroups and finite algebraic systems [5].

9.2 Maximal Subgroups

The Rees representation immediately identifies all maximal subgroups. Since

$$H_{i\lambda} \cong G,$$

group-theoretic properties can be transferred to the semigroup. For example, finiteness, solvability, commutativity, exponent, and subgroup structure of the maximal subgroups are determined by G. The outer coordinates describe how multiple isomorphic copies of G are distributed throughout the semigroup.

9.3 Matrix Representations and Semigroup Algebras

Clifford used matrix methods to represent completely simple semigroups [3]. Munn subsequently showed that semigroup algebras associated with completely simple and completely 0-simple semigroups can be described by generalized matrix algebras [7].

If K is a field and $K[G]$ is the group algebra of G, the sandwich matrix can be interpreted as a matrix over $K[G]$. For suitable matrices A and B, multiplication in the corresponding Munn algebra takes the form

$$A \circ B = APB.$$

This description reduces questions about representations, radicals, and modules of completely simple semigroup algebras to calculations involving the group algebra and the sandwich matrix.

9.4 Idempotent Generation

The explicit formula

$$e_{i\lambda} = \left(i, p_{\lambda i}^{-1}, \lambda\right)$$

makes it possible to investigate the subsemigroup generated by the idempotents. In the completely 0-simple case, the arrangement of nonzero entries in P, together with the subgroup generated by suitable products of sandwich entries, determines whether the whole semigroup or a large regular subsemigroup is idempotent-generated [8].

9.5 Rank and Generating Sets

The rank of a semigroup is the minimum cardinality of a generating set. For finite Rees matrix semigroups, rank depends on the sizes of I and Λ, the rank of the group G, and the subgroup generated by entries of a normalized sandwich matrix. Ruškuc developed rank formulas for connected completely 0-simple semigroups and obtained corresponding results for completely simple semigroups [9].

Finite generation and finite presentability can also be examined directly from the Rees data. For Rees matrix semigroups over more general base semigroups, the sizes of the index sets and the ideal generated by the sandwich entries play a decisive role [10]. Graph-theoretic methods based on nonzero entries of P have further clarified generating sets of completely 0-simple semigroups [11].

9.6 Finite Isomorphism Classification

The isomorphism criterion provides a practical classification procedure for finite completely simple semigroups:

1. Determine the isomorphism type of the maximal subgroup G.
2. Determine the numbers of R- and L-classes.
3. Normalize the sandwich matrix.
4. Apply automorphisms of G.
5. Permute rows and columns.
6. Apply permitted row and column scalings.
7. Compare the resulting matrix orbits.

This method transforms an abstract semigroup-isomorphism problem into a finite orbit problem involving group automorphisms and matrix transformations.

CONCLUSION

Rees matrix semigroups provide a complete and computationally explicit description of completely simple semigroups. Their structure is governed by three forms of information: the group G, which determines the maximal subgroups; the index sets I and Λ, which determine the minimal right and left ideals; and the sandwich matrix P, which determines the interaction among the group components.

The construction makes the main structural features immediately visible. Green's R-relation is determined by the first coordinate, the L-relation by the third coordinate, and every H-class is a maximal subgroup isomorphic to G. Idempotents and inverses have explicit formulas, while simplicity and regularity follow directly from the multiplication rule.

Normalization and the isomorphism criterion show that sandwich matrices must be classified modulo row and column permutations, group automorphisms, and group-valued scaling. The Rees–Suschkewitsch theorem then establishes that this construction is not merely an example but the complete structural model for completely simple semigroups. Its extensions and applications demonstrate the continuing importance of Rees matrix methods in finite-semigroup theory, representation theory, semigroup algebras, idempotent generation, and rank problems.

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